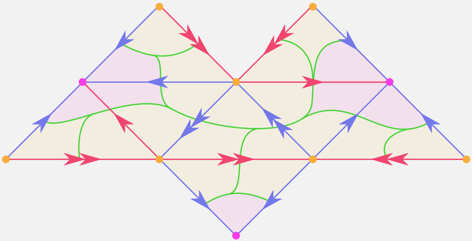
The Teichmüller polynomial via Fox calculus
(and veering triangulations)



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3-manifolds fibered over the circle

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Thurston: M is hyperbolic iff ψ is pseudo-Anosov

A pseudo-Anosov ψ has an associated stretch factor λ

Thurston norm

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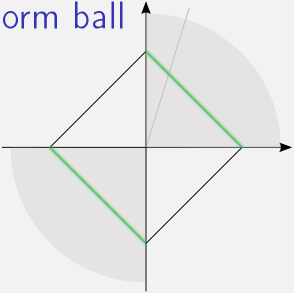
Thurston:

A norm on $H_2(M, \partial M; \mathbb{R})$ such that if α is integral then

$$\|\alpha\|_{\mathrm{Th}} = \inf \left\{ -\chi(S) \;\middle|\; S \text{ represents } \alpha \text{ and has no } S^2,\ D^2 \text{ components} \right\}$$

Fibered faces of the Thurston norm ball



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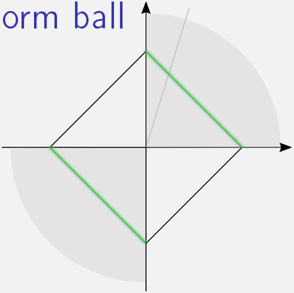

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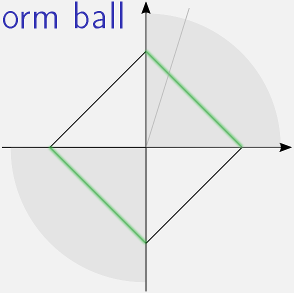

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- the homology class of the fiber of a fibration of M lies in the interior of the cone $\mathcal{C}(\mathtt{F})$ over a top-dimensional face $\mathtt{F}$ of $\mathbb{B}_{\mathrm{Th}}$
- any primitive integral class from $\mathrm{int}\ \mathcal{C}(\mathtt{F})$ can be represented by a fiber of a fibration of M over the circle

Such faces of $\mathbb{B}_{\mathrm{Th}}$ are called fibered faces

Fibrations lying over the same fibered face

If $b_1(M) > 1$ and M is fibered over the circle then it fibers in infinitely many distinct ways.

Question

How the stretch factors of different fibrations lying over the same fibered face behave?

Teichmüller polynomial

[McMullen, 1999]

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$$\Theta_{\mathrm{F}} = \sum_{h \in H} a_h \cdot h \in \mathbb{Z}[H],$$

where $a_h \in \mathbb{Z}$ and $H = H_1(M;\mathbb{Z})\,/\text{torsion}$

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$\alpha \in \mathcal{C}(\mathrm{F}) \cap H_2(M, \partial M; \mathbb{Z})$ primitive

the stretch factor of the monodromy of the fibration determined by

α is equal to the largest real root of

$$\Theta_{\mathrm{F}}^{\alpha}(z) = \sum_{h \in H} a_h \cdot z^{\langle \alpha, h \rangle}$$

Teichmüller polynomial – applications and computation

- can be computed using just one fibration
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Computation

- McMullen's original algorithm general but hard to implement
- simpler algorithms that works in some special cases: Lanneau-Valdez 2017, Baik-Wu-Kim-Jo 2020, Billet-Lechti 2019
- Landry-Minsky-Taylor 2019: the **taut polynomial** which generalizes the Teichmüller polynomial

Veering triangulations

special class of ideal triangulations of cusped 3-manifolds which

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- **Fried:** every fibered face of the Thurston norm ball has a canonical (up to isotopy and reparametrization) circular pseudo-Anosov flow associated to it
- The Teichmüller polynomial is really an invariant of the associated pseudo-Anosov flow

Taut polynomial

[Landry-Minsky-Taylor, 2019]

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F – fibered face of the Thurston norm ball in $H_2(M, \partial M)$

$\Lambda = \{\ell_1, \ldots, \ell_k\}$ – singular orbits of the associated flow

Agol, 2010: $M^\circ = M - \Lambda$ has a veering triangulation $\mathcal{V}$ which encodes the flow associated to F

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$i : M^\circ \hookrightarrow M \rightsquigarrow i_* : H_1(M^\circ; \mathbb{Z})/\text{torsion} \to H_1(M; \mathbb{Z})/\text{torsion}$

Landry-Minsky-Taylor, 2019: $\Theta_{\mathrm{F}} = i_*(\Theta_{\mathcal{V}})$

Computing the taut polynomial

n – number of tetrahedra of $\mathcal{V}$

LMT's original definition: $\Theta_{\mathcal{V}}$ is the gcd of the maximal minors of a $n \times 2n$ matrix with coefficients in $\mathbb{Z}[H^\circ]$

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P., 2020: Algorithm to compute $\Theta_{\mathcal{V}}$ for any $\mathcal{V}$

It is enough to compute $n+1$ minors

Computing the Alexander polynomial

Alexander polynomial Δ_M of M can be computed

- using a triangulation of M:

  $n = \#$ tetrahedra

  have to compute the gcd of $n+1$ minors of dimension n

- from any presentation of $\pi_1(M)$ using Fox calculus

Fox calculus

Presentation $\pi_1(M) = \langle S \mid R \rangle \rightsquigarrow$ matrix $J \in \mathbb{Z}[\pi_1(M)]^{|S|\times|R|}$

(Jacobian matrix / Alexander matrix / Fox derivative)

Property of J: mapping its entries through a $\pi_1(M) \to H_1(M;\mathbb{Z})/\text{torsion}$ gives a matrix whose gcd of the $|R| \times |R|$ minors is Δ_M.

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Major advantage: The fundamental group of a 3-manifold given as a triangulation with n tetrahedra typically has a presentation with (much) less than n relations.

Fox calculus and twisted Alexander polynomials

Can consider different representations of $\pi_1(M)$, eg.

$\pi : \pi_1(M) \to H_1(M; \mathbb{Z})/\text{torsion}$

$\omega : \pi_1(M) \to \mathbb{Z}/2 = \{-1, 1\}$

tensor representation:

$$\omega \otimes \pi : \pi_1(M) \to H_1(M; \mathbb{Z})/\text{torsion}$$

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the gcd of the $|R| \times |R|$ minors of $(\omega \otimes \pi)(J)$ gives a twisted Alexander polynomial $\Delta_M^{\omega\otimes\pi}$

P., 2021

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Consequences:

- Quick computation of the taut polynomial, and hence the Teichmüller polynomial, using Fox calculus.

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- For any M only finitely many candidates for the taut polynomial.

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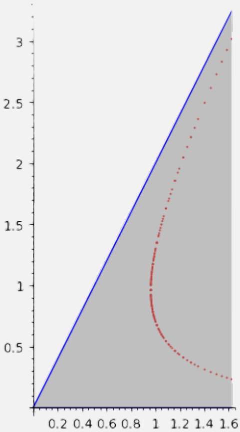
- For any M only finitely many candidates for the taut polynomial. (No sufficient condition for the existence of a veering triangulation on M is known, but we know what are the possible taut polynomials!)
- Algebraic properties of the taut polynomial = algebraic properties of twisted Alexander polynomials.

These computations are implemented (Veering GitHub), so we can quickly draw pictures like this:



fibered class α $\rightsquigarrow$ stretch factor λ

$\rightsquigarrow$ entropy $\log(\lambda)$

entropy for $k \cdot \alpha$ is $\frac{\log(\lambda)}{k}$

plotted:

unique point on the ray through α with

entropy = 1

(i.e. $\log(\lambda) \cdot \alpha$)